Economic growth, habit formation, and business cycle

Abstract

The purpose of this paper is to demonstrate the existence of business cycles in a theoretical economic growth model with endogenous habit formation, wealth, and preference change proposed. The model studies preference changes and habit formation in an extended Solow-Uzawa neoclassical development model with elastic labor supply. The study introduces the preference change based on the traditional literature on time preference and habit formation in the Ramsey approach to modelling behavior of households. It makes the model proposed by Zhang (2013) more robust with all the time-independent parameters generalized as time-dependent parameters. The model is simulated. Business cycles and preference oscillations are demonstrated under various exogenous periodic shocks.

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1 Introduction

The main purpose of this study examines the existence of business cycles in a formal development model proposed by Zhang (2013). The model includes habit formation and preference change as endogenous variables of economic growth. This study makes the original model more robust by generalizing all the time-independent parameters as time-dependent parameters. We show how various exogenous changes affect the entire economic system. There are many studies in theoretical research and empirical approach about causes and effects of business cycles (e.g., Zhang, 1991, 2005, 2006; Chatterjee and Ravikumar, 1992; Gabaix, 2011; Giovanni, et al. 2014; Stella, 2015; Choudhry, et al., 2016; Ohtsuka, 2018; Carvalho and Grassi, 2019; Khan, et al., 2019; and Kiyotaki and Moore, 2019). But one can find only a few theoretical models which show periodic fluctuations caused by dynamic

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interactions among wealth accumulation, economic structural change, labor supply, habit formation, and preference change. This study contributes the literature of economic cycles by showing the existence of exogenous business cycles in a compact analytical framework.

Some empirical studies demonstrate dynamic interactions among preference changes, economic structures, and growth (e.g., Fuchs, 1982; Horioka, 1990; Olsen, 1993; Benjamin, et al., 2012; Sheldon, 1997; 1998; Becker and Mulligan, 1997; Kirby et al. 2002; and Chao et al., 2009; Liefooghe, 2017, and Urcelay and Jonkman, 2019). This study examines similar issues but in alternative approach. We model preference changes introduced by Zhang (2013). Zhang’s approach is based on the Ramsey approach to modelling time preference change and habit formation. It is argued that “Time preference plays a fundamental role in theories of saving and investment, economic growth, interest rate determination and asset pricing, addiction, and many other issues that are getting increasing attention from economists. Yet, since Samuelson’s [1937] discounted utility model, rates of time preference are almost invariably taken as “given” or exogenous, with little discussion of what determines their level.” (Becker and Mulligan, 1997: 729). One finds an early mathematical approach to endogenous time preference by Uzawa (1968). Since then, various studies with formal modelling are carried out. Possible implications of endogenous time preference for dynamic economics are investigated with different approaches (e.g., Epstein and Hynes, 1983; Lucas and Stokey, 1984; Epstein, 1987; Obstfeld, 1990; Shin and Epstein, 1993; Palivos et al. 1997; Druegon, 1996, 2000; Stern, 2006; Meng, 2006; and Dioikitopoulos and Kalyvitis, 2010; Galor and Ozak, 2016; Wang, et al., 2016; Hayakawa, 2020; and Jordan, et al., 2020). These researches demonstrate that economic growth models with endogenous time preference can explain economic growth and development better than the traditional models with the variables fixed.

Zhang (2013) uses main ideas in the theoretical literature to model preference changes. Zhang’s model on modelling habit formation and habit persistence was influenced by the formal economic analysis in Duesenberry (1949). The basic idea implies that consumers may be used to a given “standard of living”. The idea of habit formation is found useful in explaining economic issues (e.g., Pollak, 1970; Mehra and Prescott, 1985; Sundaresan, 1989; Abell, 1990; Constantinides, 1990; Becker, 1992; de la Croix, 1996; Campbell and Cochrane, 1999; Boldrin et al. 2001; Ravn et al. 2006; Huang, 2012; Carden and Wood, 2018; Bouche and Miguel, 2019; and Tadajewski, 2019). This study uses this idea to examine the dynamics of the preferences for consumer goods and leisure. It applies Uzawa’s analytical framework for economic structure (Uzawa, 1961). This framework is applied by many other economists (e.g., Diamond, 1965; Stiglitz, 1967; Mino, 1996; and Druegon and Venditti, 2001). The approach to consumer behaviour follows Zhang’s alternative model (Zhang, 1993). The rest of the paper is structured in 4 sections. Section 2 develops the theoretical model in which labor supply and preference are endogenous. Section 3 shows the dynamic properties of the model and follows the movement of the economic system with all the parameters time dependent. Section 4 gives comparative dynamic analysis with regards to different exogenous periodic shocks to the system. Section 5 concludes the paper.
2 The Formal Model with Habit Formation

This section proposes a model in neoclassical growth analytical framework. The modeling of producers, economic structure, and technology follows the standard two sectors growth model (Solow, 1956; Uzawa, 1961; Burmeister and Dobell 1970; and Barro and Sala-i-Martin, 1995). The capital of the economy is assumed to be owned solely by the households. All the markets are perfectly competitive. There are two homogenous factor inputs, capital and labour, which are smoothly substitutable each other in all the sectors. The input factors are freely transferable between the sectors without any cost and loss. Capital depreciates at an exponential rate $\delta_k(t)$. We select the capital goods as numeraire.

The production sectors

Let subscripts $i$ and $s$ stand for, respectively the capital goods and the consumption goods sectors. We use $F_j(t)$ to denote the output of sector $j$. The variables $K_j(t)$ and $N_j(t)$ represent, respectively, sector $j$’s capital and labour inputs. We take the production functions on the following form:

$$F_j(t) = A_j K_j^{\alpha_j(t)}(t) N_j^{\beta_j(t)}(t), \alpha_j(t), \beta_j(t) > 0, \alpha_j(t) + \beta_j(t) = 1, j = i, s, \tag{1}$$

where $A_j(t)$ is sector $j$’s total factor productivity, $\alpha_j(t)$ and $\beta_j(t)$ are, respectively, the sector’s output elasticities of capital and labour. The production functions can be expressed as follows:

$$f_j(k_j(t), t) = A_j(t) k_j^{\alpha_j(t)}(t), A_j(t) > 0, 0 < \alpha_j(t) < 1, j = i, s, \tag{2}$$

where

$$f_j(t) = f_j\left(k_j(t)\right), f_j(t) = \frac{F_j(t)}{N_j(t)}, k_j(t) = \frac{K_j(t)}{N_j(t)}, \ j = i, s.$$

Let $p(t)$ stand for the price of consumption goods. The perfect competition implies that the input factors earn their marginal products. The marginal conditions are

$$r(t) + \delta_k(t) = \alpha_i(t) A_i(t) k_i^{\beta_i(t)}(t) = \alpha_s(t) A_s(t) p(t) k_s^{-\beta_j(t)}(t),$$

$$w(t) = \beta_i(t) A_i(t) k_i^{\alpha_i(t)}(t) = \beta_s(t) A_s(t) p(t) k_s^{\alpha_j(t)}(t). \tag{3}$$

where interest rate of interest $r(t)$ and wage rate $w(t)$ are decided in perfect competitive markets.

Input factors being fully employed

The full employment conditions are:

$$K_i(t) + K_s(t) = K(t), N_i(t) + N_s(t) = N(t), \tag{4}$$

where $N(t) = T(t) N_0(t)$ is the total labour supply, $N_0(t)$ is the population, and $T(t)$ is work time of the representative household. We rewrite (4) as:

$$n_i(t) k_i(t) + n_s(t) k_s(t) = k(t), n_i(t) + n_s(t) = 1, \tag{5}$$

in which

$$k(t) = \frac{K(t)}{N(t)}, n_j(t) = \frac{N_j(t)}{N(t)}, j = i, s. \tag{6}$$
Households’ disposable income and budget constraint

We model the household by the approach by Zhang (1993). In this approach, the disposable income consists of the current income and the value of wealth which is available for consumption and saving. The current income \( y(t) \) is the sum of the interest payment \( r(t)\dot{k}(t) \) (where \( \dot{k}(t) \equiv K(t)/N_0(t) \)) and the wage payment \( w(t)T(t) \). That is:

\[
y(t) = r(t) \dot{k}(t) + w(t) T(t).
\]

The disposable income is thus given by:

\[
\hat{y}(t) = y(t) + \dot{k}(t) = (1 + r(t))\dot{k}(t) + w(t) T(t). \tag{7}
\]

The disposable income is distributed between consumption \( c(t) \) and saving \( s(t) \). We have the budget constraint as follows:

\[
p(t) c(t) + s(t) = \hat{y}(t). \tag{8}
\]

We use \( T_h(t) \) to represent the leisure time. The total time for work and leisure is fixed and \( T_0 \). The household’s time distribution satisfies:

\[
T(t) + T_h(t) = T_0. \tag{9}
\]

Insert (9) in (8):

\[
w(t) T_h(t) + p(t) c(t) + s(t) = \hat{y}(t) \equiv r(t) \dot{k}(t) + w(t) T_0 + \dot{k}(t). \tag{10}
\]

Households’ optimal decision

The utility \( U(t) \) is related to \( T_h(t), c(t), \) and \( s(t) \) as follows:

\[
U(t) = T_h^{\sigma_0(t)}(t) c^{\xi_0(t)}(t) s^{\lambda_0(t)}(t), \sigma_0(t), \xi_0(t), \lambda_0(t) > 0,
\]

Where \( \xi_0(t) \) the propensity to consume, \( \lambda_0(t) \) the propensity to own wealth, and \( \sigma_0(t) \) is the propensity to use leisure time. The maximization of \( U(t) \) subject to (10) implies:

\[
w(t) T_h(t) = \sigma(t) \hat{y}(t), p(t) c(t) = \xi(t) \hat{y}(t), s(t) = \lambda(t) \hat{y}(t), \tag{11}
\]

where

\[
\sigma(t) \equiv p(t) \sigma_0(t), \xi(t) \equiv p(t) \xi_0(t), \lambda(t) \equiv p(t) \lambda_0(t), \rho(t) \equiv \frac{1}{\sigma_0(t) + \xi_0(t) + \lambda_0(t)}.
\]

Wealth accumulation

The change in the household’s wealth is the saving minus the dissaving:

\[
\dot{k}(t) = s(t) - \dot{k}(t) = \lambda(t)\hat{y}(t) - \dot{k}(t) - \frac{N_0(t)}{N_0(t)}\dot{k}(t). \tag{12}
\]

Demand and supply of consumer goods

The balance condition in the consumer goods market is:

\[
c(t) N_0(t) = F_0(t). \tag{13}
\]
Demand and supply of capital goods

The output of the capital goods sector is used up for the net savings and the capital depreciation:

\[ S(t) - K(t) + \delta_k(t)K(t) = F_i(t), (14) \]

where \( S(t) - K(t) \) is the net saving and \( \delta_k(t)K(t) \) is the capital depreciation.

The time preference and the propensity to save

Uzawa (1968) initiated the formal modelling of preference change in the Ramsey model (see also Blanchard and Fischer, 1989; Das, 2003; and Hirose and Ikeda, 2008; Chang et al, 2011). According Persson and Svensson (1985: 45), Uzawa’s idea is “arbitrary and even counterintuitive”. They argue that the traditional approach is against the evidence of savings as decreasing function of real wealth. There are studies in which the rate of time preference rises in real wealth (e.g., Dornbusch and Frenkel, 1973; Orphanides and Solow, 1990; Smithin, 2004; and Kam and Mohsin, 2006). Zhang (2013) assumes the propensity to save to be related to the real wealth and wage rate in a linear relation:

\[ \lambda_0(t) = \bar{\lambda}(t) + \lambda_w(t)w(t) + \lambda_k(t)\bar{k}(t), (15) \]

where \( \bar{\lambda}(t) > 0, \lambda_w(t), \) and \( \lambda_k(t) \) are parameters. It should be noted that in Zhang (2013) \( \bar{\lambda}, \lambda_w, \) and \( \lambda_k \) are variant in time.

The habit formation and the propensity to consume consumer goods

To illustrate the approach in the literature about habit formation, we cite infinitely lived representative consumer in Corrado and Holly (2011). The representative household’s expected utility is as follows (see also Carroll, 2000; Amano and Laubach, 2004; Deaton and Muellbauer, 1980):

\[ U = E_t \left\{ \sum_{j=1}^{\infty} \beta^j U_{t+j}(\cdot) \right\}, \]

where \( U_{t+j}(\cdot) \) is the instantaneous utility function, \( \beta = 1/(1 + \theta) \) is the parameter of impatience, and \( \theta \) represents the subjective rate of time preference. They use the following utility function:

\[ U_t = \frac{(C_t h_t^{-v})^{1-\alpha}}{1 - \alpha}, \]

where \( C_t \) is the consumption level, \( h_t \) is the stock of habit of the consumption good, and \( \alpha \) is a parameter. The parameter \( v \) implies the importance of the habit stock. Alvarez-Cuadrado et al. (2004) give a habit formation as:

\[ h(t) = \rho \int_{-\infty}^{t} e^{h_0(s-t)} C^\varphi(s)\hat{C}^{1-\varphi}(s) ds, \rho > 0, 0 \leq \varphi \leq 1, \]

where \( C(t) \) stands for the consumption level and \( \hat{C}(t) \) the economy-wide average consumption level (see also Lusardi, 1996; Storesletten, et al. 2004; Gómez, 2008; and Lise and
Seitz, 2011). A larger value for \( h_0 \) means that lower weights is given to more distant values of the levels of consumption. Taking derivatives of the above equation in time yields:

\[
\dot{h}(t) = h_0[C^\varphi(s)\tilde{c}^{1-\varphi(s)} - h(t)].
\]

Being influenced by the above formation, we construct the habit formation of consumer goods as follows:

\[
\dot{h}_c(t) = \dot{\xi}(t)[c(t) - h_c(t)]. \quad (16)
\]

A higher level of the current consumption than the level of the habit stock implies a rise in the level of habit stock, and vice versa. As in Zhang (2013), we specify the propensity to consume as follows:

\[
\xi_0(t) = \dot{\xi}(t) + \xi_w(t) w(t) + \xi_h(t) h_c(t), \quad (17)
\]

where \( \dot{\xi}(t) > 0, \xi_w(t) \) and \( \xi_h(t) \geq 0 \) are parameters.

The habit formation and the propensity to have leisure time

We model endogenous propensity to use leisure like the change in the propensity to consume consumer goods. Like (16), we have the differential equation for the habit stock of the leisure time \( h_T(t) \) as:

\[
\dot{h}_T(t) = \dot{\sigma}(t)[T_h(t) - h_T(t)], \quad (18)
\]

where \( \dot{\sigma}(t) \) is a parameter for the relative weights of leisure time. We relate the propensity to use leisure time to the other variables as:

\[
\sigma_0(t) = \dot{\sigma}(t) + \sigma_w(t) w(t) + \sigma_h(t) h_T(t), \quad (19)
\]

where the parameters, \( \dot{\sigma}(t) \) is positive, the sign of \( \sigma_w(t) \) is ambiguous, and \( \sigma_h(t) \geq 0 \) is usually nonnegative.

We completed generalizing the model by Zhang (2013) by allowing all the constant parameters to variant in time. This implies that the original model becomes far more robust. The new model can deal with effects of any type of continuous exogenous shocks on the motion of the economic system. We now study the behavior of the model.

3 The Motion of the Economic System

The appendix shows that we can describe the movement of the national economy with three nonlinear differential equations with \( k_i(t), h_c(t), h_T(t) \), and \( t \) as variables. The following lemma gives a procedure for computing the movement of all the variables.

Lemma

The motion of the national economy is given by three nonlinear differential equations with \( k_i(t), h_c(t), h_T(t) \), and \( t \) as the variables:

\[
\dot{k}_i(t) = \Lambda_k(k_i(t), h_c(t), h_T(t), t),
\]

\[
\dot{h}_c(t) = \Lambda_c(k_i(t), h_c(t), h_T(t), t),
\]

\[
\dot{h}_T(t) = \Lambda_T(k_i(t), h_c(t), h_T(t), t), \quad (20)
\]
in which \( \Lambda_k, \Lambda_c, \) and \( \Lambda_T \) are functions of \( k_i(t), h_c(t), h_T(t), \) and \( t \) given in the appendix. All the other variables are related to \( k_i(t), h_c(t) \) and \( h_T(t) \) as follows: \( k_s(t) = ak_i(t) \to \tau(t) \) and \( w(t) \) by (3) \( \to p(t) \) by (A2) \( \to \sigma_0(t) \) by (19) \( \to \xi_0(t) \) by (17) \( \to \lambda_0(t) \) by (A14) \( \to \lambda(t), \xi(t), \) and \( \sigma(t) \) by (11) \( \to k(t) \) by (A8) \( \to n_i(t) \) and \( n_s(t) \) by (A3) \( \to T(t) \) by (A9) \( \to \tilde{k}(t) = k(t)T(t) \to T_N(t) = T_0(t) - T(t) \to N(t) = T(t)N_0 \to N_i(t) = n_i(t)N(t) \to N_s(t) = n_s(t)N(t) \to K_i(t) = k_i(t)N_i(t) \to K_s(t) = k_s(t)N_s(t) \to F_j(t) \) by (1) \( \to f_j(t) = F_j(t)/N_j(t) \to \bar{y}(t) \) by (A5) \( \to c(t) \) and \( s(t) \) by (11).

We solve \( k_i(t), h_c(t), \) and \( h_T(t) \) with (20). By following the procedure, we get the values of all the variables as functions of \( k_i(t), h_c(t), h_T(t), \) and \( t \). We now simulate the model. First, let us consider that that all the parameters are invariant in time as follows:

\[
T_0 = 1, N_0 = 10, A_i = 1.1, A_s = 0.9, \alpha_i = 0.35, \alpha_s = 0.3, \bar{\lambda} = 0.5, \lambda_w = -0.01, \lambda_k = 0.02, \\
\bar{\xi} = 0.1, \bar{\xi} = 0.25, \bar{\xi}_y = 0.01, \xi_h = 0.02, \bar{\sigma} = 0.4, \bar{\sigma} = 0.25, \sigma_w = -0.01, \sigma_h = 0.01. \quad (21)
\]

We have the depreciation rate fixed at \( \delta_k = 0.03 \) in this section. We now specify the initial conditions as follows:

\[
h_c(0) = 0.4, h_T(0) = 0.6, k_i(0) = 3.7.
\]

The rest of the results in this section is obtained in Zhang (2013). Figure 1 plots the simulation result.

Insert Figure 1 here.

The equilibrium values of the variables are calculated as follows:

\[
F = 7.20, k = 2.87, F_i = 0.39, F_s = 5.26, N = 4.503, N_i = 0.225, N_s = 4.277, K = 12.90, \\
K_i = 0.80, K_s = 12.10, r = 0.14, p = 1.30, w = 1.11, \lambda_0 = 0.52, \xi_0 = 0.27, \sigma_0 = 0.24, \\
T_h = h_T = 0.55, \bar{k} = 1.29, c = h_c = 0.23.
\]

The three eigenvalues are given by: \( \{-0.39, -0.28, -0.10\} \). The unique equilibrium is thus stable.

4 Comparative Dynamic Analysis with Exogenous Oscillatory Shocks

The previous section gives a unique stable equilibrium. The stability implies the validity of comparative dynamic analysis. We now study the impact of exogenous oscillations of parameters on the national economy. We consider Figure 1 as the long-term trend values. This section shows how the system moves over time when it experiences some exogenous oscillatory shocks.

Fluctuations in parameter \( \xi_h \)

We now consider the effect of the habit stock of consumer goods on the propensity to consume to experience the following fluctuations:

\[
\xi_h(t) = 0.02 + 0.005 \sin(t).
\]
Figure 2 shows the simulation. As the habit stock oscillatorily affects the propensity to consume, the habit stock is fluctuated. The fluctuations cause the propensity to consume to be oscillatorily. The fluctuations in the propensity to consume make propensities to have leisure and to own wealth oscillate. The consumption level fluctuates around the long-term trend value. We see that all the variables are oscillatorily affected by the exogenous fluctuations. The national output, the total labor force, total wealth, the work hours, the output levels, and inputs of the two sectors are fluctuated.

**Insert Figure 2 here.**

**Fluctuations in parameter $\sigma_w$**

We examine the effect of the wage rate on the propensity to have leisure to be exogenously perturbed as follows:

$$\sigma_w(t) = -0.01 + 0.01 \sin(t).$$

Figure 3 shows the results. As the effect of the wage rate on the propensity to use leisure time oscillates, all the variables are affected. Both the habit stock of leisure time and leisure time fluctuate. The propensity to have leisure, the propensity to save and the propensity to consume are strongly affected. The wealth, work time and consumption level are periodically changed. The price, wage rate and interest rate fluctuate. The national output, total wealth, and total labor force are also changed periodically.

**Insert Figure 3 here.**

**Fluctuations in effect of wealth on the propensity to save**

Economists are concerned with possible effects of changes in the propensity to save on development and economic structure. It is well known that Adam Smith and Keynes have different points of view on effects. Adam Smith argued that a rise in the propensity to save encourages economic development, but Keynes argues for the opposite effect. We now allow the effect of the effect of wealth on the propensity to save to be exogenously changed as follows:

$$\lambda_k(t) = 0.02 + 0.01 \sin(t).$$

Figure 4 shows the simulation. All the variables are affected. The propensity to save is fluctuated. In association with the exogenous fluctuations the propensity to have leisure fluctuates. The fluctuations in the preference result in slight oscillations in the habit stock of leisure time and in the habit stock of consumer goods.

**Insert Figure 4 here.**

**Fluctuations in the depreciation rate of physical capital**

In the literature on business cycles there are only a few studies on effects of fluctuations of depreciation rates of capital on economic systems. We now study what will happen in the economic system if the capital depreciation rate is periodically changed as follows:

$$\delta_k(t) = 0.03 + 0.01 \sin(t).$$

Figure 5 shows the simulation. The fluctuations in the capital depreciation rate affect all the variables. There are periodic changes in the total wealth, the output levels and two input factors...
Economic growth, habit formation, and business cycle

of the two sectors. The fluctuations in the parameter also lead to oscillations in the interest rate, wage rate, and the price of consumer goods.

Insert Figure 5 here.

**Oscillations in the total factor productivity of the capital goods sector**

We introduce exogenous oscillations in $A_t$ as follows:

$$A_t(t) = 1.1 + 0.05 \sin(t).$$

Figure 6 shows the simulation result. The changes in the total productivity affect all the variables. There are periodic changes in the output levels, the total wealth, and the input factors of the sectors. The national economy also experiences periodic changes in the price of consumer goods, the interest rate, the wage rate. The propensities to save, to have leisure and to consume consumer goods fluctuate.

Insert Figure 6 here.

5 Concluding Remarks

This paper studied business cycles in Zhang’s dynamic model with endogenous wealth, preference change, and habit formation. The habit formation and preference changes are modelled in the Solow-Uzawa growth model with endogenous labor supply. It describes dynamics of economic structure with habit formation, preference change, and wealth accumulation. The paper generalized Zhang’s model by treating all constant parameters as parameters variant in time. The model was modelled. Business cycles are caused by various types of exogenous periodic shocks. As the production functions and utility function of the Cobb-Douglas form, we can examine the model when the economy has different types of technology and preference. There are numerous studies on generating and extending the Solow-Uzawa growth model. We can also generalize the model in this study on the basis of the literature.

Appendix 1: Proving Lemma

We now check the Lemma. From (3), we have:

$$k_s = \alpha k_i, (A1)$$

where

$$\alpha \equiv \frac{\beta_i \alpha_s}{\beta_s \alpha_i}, \beta_j \equiv 1 - \alpha_j, j = i, s.$$  

By $k_s = \alpha k_i$ and $\beta_i f_i = \beta_s f_s$, we solve:

$$p = p_0 k_i^{\alpha_i - \alpha_s}, (A2)$$

where $p_0 \equiv \beta_i A_i / \beta_s \alpha^s A_s$. In this study, we require $\alpha_i \neq \alpha_s$. From (5) we solve:

$$n_i = \frac{\tilde{\beta}(\alpha k_i - k)}{k_i}, n_s = \frac{\tilde{\beta}(k - k_i)}{k_i}, (A3)$$
where \( \tilde{\beta} \equiv 1/(\alpha - 1) \). By the definitions of \( S, K, s, \) and \( k \), we have:

\[
S - \delta K = (s - \delta \tilde{k})N_0, \quad \text{(A4)}
\]

where \( \delta \equiv 1 - \delta_k \). From (A4), (11) and (14), we obtain:

\[
\hat{y} = \frac{n_i T f_i + \delta \tilde{k}}{\lambda}. \quad \text{(A5)}
\]

From (A5), \( wT_0 - wT = \sigma \hat{y} \), and \( \tilde{k} = kT \), we solve:

\[
T = \frac{\lambda_0 w T_0}{\sigma_0 n_i f_i + \sigma_0 \delta k + \lambda_0 w}. \quad \text{(A6)}
\]

From \( pc = \xi \hat{y}, c = n_s T f_s, \) and \( p = f'_s / f'_s \), we get \( \hat{y} = T n_s A_s \alpha^s p_0 k^a_i / \xi \). From this equation and (A5), we have:

\[
\frac{(n_i A_i k^a_i + \delta k) \xi_0}{\lambda_0} = \frac{\beta_i A_i}{\beta_s} n_s k^a_i, \quad \text{(A7)}
\]

in which we apply \( \tilde{k} = kT \). Insert (A2) in (A7):

\[
k(k_i, \lambda_0, \xi_0) = \frac{(\alpha + \xi_\lambda \lambda_0) k_i}{f + \xi_\lambda \lambda_0}, \quad \text{(A8)}
\]

where

\[
\xi_\lambda(\xi_0) \equiv \frac{\beta_i}{\beta_s} \xi_0, \quad \bar{\alpha} \equiv (1 - \alpha) \frac{\delta}{A_i}, \quad \tilde{f}(k_i) \equiv 1 + \bar{\alpha} k_i. \]

From (11), we have:

\[
\frac{w T_h}{p c} = \frac{\sigma_0}{\xi_0}
\]

Insert \( T_h = T_0 - T \) in the above equation:

\[
T = (n_s \bar{\sigma} + 1)^{-1} T_0, \quad \text{(A9)}
\]

where we also use \( c = n_s T f_s \) and \( \bar{\sigma}(k_i, \xi_0) \equiv f_s p \sigma_0 / w_0 \xi_0 \). Insert \( \tilde{k} = kT \) in (15):

\[
\lambda_0 = \bar{\lambda} + \lambda_k w + \lambda_k k T. \quad \text{(A10)}
\]

Insert (A9) in (A10)

\[
\lambda_0 (n_s \bar{\sigma} + 1) = n_s \bar{\sigma} (\bar{\lambda} + \lambda_k w) + \bar{\lambda} + \lambda_k w + \lambda_k k T_0. \quad \text{(A11)}
\]

Insert \( n_s = (k/k_i - 1)\tilde{\beta} \) in (A11)

\[
(1 - \tilde{\beta} \bar{\sigma}) \lambda_0 = \left[ \frac{\beta_i \bar{\sigma}_1}{k_i} + \lambda_k T_0 - \frac{\tilde{\beta} \bar{\sigma} \lambda_0}{k_i} \right] k + \bar{\sigma}_0,
\]

\[
\text{(A12)}
\]

where

\[
\tilde{\beta} \equiv \frac{1}{\alpha - 1}, \quad \bar{\sigma}_1(k_i, \xi_0) \equiv (\bar{\lambda} + \lambda_k w) \bar{\sigma}, \quad \bar{\sigma}_0 (k_i, \xi_0) \equiv (1 - \tilde{\beta} \bar{\sigma})(\bar{\lambda} + \lambda_k w).
\]

Insert (A8) in (A12)

\[
\lambda^2_0 + g_1 \lambda_0 - g_2 = 0, \quad \text{(A13)}
\]

in which
\[ g_1(k_i, \xi_0) \equiv \frac{(1 - \beta \tilde{\sigma}) f - \xi_0 \bar{a}_0 + \alpha \beta \tilde{\sigma} - (\beta \tilde{\sigma}_i + \lambda_k k_i T_0) \xi_0}{g_0}, \]
\[ g_2(k_i, \xi_0) \equiv \frac{\tilde{f} \bar{a}_0 + (\beta \tilde{\sigma}_i + \lambda_k k_i T_0) \alpha}{g_0}, \]
\[ g_0(k_i, \xi_0) \equiv \xi_0. \]

We solve (A13)
\[ \lambda_0(k_i, \xi_0) = \frac{-g_1 \pm \sqrt{g_1^2 + 4 g_2}}{2}. \] (A14)

Equation (A13) has two solutions. With the parameter values in the simulation, we have the unique meaningful solution:
\[ \lambda_0(k_i, \xi_0) = \frac{-g_1 + \sqrt{g_1^2 + 4 g_2}}{2}. \]

With the procedure in the Lemma all the variables are given as functions of \( k_i, h_c, h_T, \) and \( t. \) We denote the function for wealth by: \( \tilde{k} = \varphi(k_i, h_c, h_T, t). \) From (14), (16) and (18), we have
\[ \tilde{k} = \tilde{\Lambda}(k_i, h_c, h_T, t) \equiv \lambda \bar{y} - \tilde{k}, \] (A15)
\[ h_c = \Lambda_c(k_i, h_c, h_T, t) \equiv \bar{c} - h_c, \]
\[ h_T = \Lambda_T(k_i, h_c, h_T, t) \equiv \bar{T}(h_h - h_T). \] (A16)

Taking derivatives of \( \tilde{k} = \varphi(k_i, h_c, h_T, t) \) with respect to time yields
\[ \dot{\tilde{k}} = \frac{\partial \varphi}{\partial k_i} \dot{k}_i + \frac{\partial \varphi}{\partial t} + \Lambda_c \frac{\partial \varphi}{\partial h_c} + \Lambda_T \frac{\partial \varphi}{\partial h_T}, \] (A17)

in which we also apply (A17). The tedious expressions are not provided here. From (A15) and (A17), we have:
\[ \dot{k}_i = \Lambda_k(k_i, h_c, h_T, t) \equiv \left( \bar{A} - \frac{\partial \varphi}{\partial t} - \Lambda_c \frac{\partial \varphi}{\partial h_c} - \Lambda_T \frac{\partial \varphi}{\partial h_T} \right) \frac{\partial \varphi}{\partial k_i}^{-1}. \] (A18)

The Lemma was checked.

References


**Appendix 2**

![Image](image_url)

**Figure 1. The Motion of the National Economy**
Figure 2. Fluctuations in the Effect of the Habit Stock of Consumer Goods

Figure 3. Fluctuations in Effects of Wage Rate on the Propensity to Use Leisure Time
Figure 4. Fluctuations in Effect of Wealth on the Propensity to Save

Figure 5. Fluctuations in the Depreciation Rate of Physical Capital
Figure 6. Oscillations in the Total Factor Productivity of the Capital Goods Sector